A LLT-like test for proving the primality of Fermat numbers.

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In 1876, Édouard Lucas discovered a method for proving that a number is prime or composite without searching its factors. His method was based on the properties of the Lucas Sequences. He first used his method for Mersenne numbers and proved that $2^{127} - 1$ is a prime. In 1930, Derrick Lehmer provided a complete and clean proof. This test of primality for Mersenne numbers is now known as: Lucas-Lehmer Test (LLT).

Few people know that Lucas also used his method for proving that a Fermat number is prime or composite, still with an unclear proof. He used his method for proving that $2^{2^n} + 1$ is composite. Lehmer did not provide a proof of Lucas’ method for Fermat numbers.

This paper provides a proof of a LLT-like test for Fermat numbers, based on the properties of Lucas Sequences and based on the method of Lehmer. The seed (the starting value $S_0$ of the $\{S_i\}$ sequence) used here is 5, though Lucas used 6. In 1960, Kustaa Inkeri provided a full proof with seed 8.

Primality tests for special numbers are classified into $N - 1$ and $N + 1$ categories, meaning that the numbers $N - 1$ or $N + 1$ can be completely or partially factored. Since many books talk about the LLT only in the $N + 1$ chapter for Mersenne numbers $N = 2^q - 1$, it seemed useful to remind that the LLT can also be used for numbers $N$ such that $N - 1$ is easy to factor, like Fermat numbers $N = 2^{2^n} + 1$, by providing a proof à la Lehmer.

**Theorem 1**

$F_n = 2^{2^n} + 1$ ($n \geq 1$) is a prime if and only if it divides $S_{2^n-2}$, where $S_0 = 5$ and $S_i = S_{i-1}^2 - 2$ for $i = 1, 2, 3, \ldots 2^n - 2$.

The proof is based on chapters 4 (The Lucas Functions) and 8.4 (The Lehmer Functions) of the book ”Édouard Lucas and Primality Testing” of H. C. Williams (A Wiley-Interscience publication, 1998).

Chapter 1 explains how the $(P, Q)$ parameters have been found. Then Chapter 2 provides the Lehmer theorems used for the proof. Then Chapter 3 and 4 provide the proof for: $F_n$ prime $\Rightarrow F_n \mid S_{2^n-2}$ and the converse, proving theorem 1. Chapter 5 provides numerical examples. The appendix in Chapter 6 provides first values of $U_n$ and $V_n$ plus some properties.

**AMS Classification:** 11A51 (Primality), 11B39 (Lucas Sequences), 11-03 (Historical), 01A55 (19th century), 01A60 (20th century).
1 Lucas Sequence with $P = \sqrt{R}$

Let $S_0 = 5$ and $S_i = S_{i-1}^2 - 2 \cdot S_i = 23$, $S_2 = 527 = 17 \times 31$, ...

It has been checked that: \[ S_{2n-2} \equiv 0 \pmod{F_n} \text{ for } n = 1, \ldots, 4 \]
\[ S_{2n-2} \not\equiv 0 \pmod{F_n} \text{ for } n = 5, \ldots, 14 \]

Hereafter, we search a Lucas Sequence $(U_m)_{m \geq 0}$ and its companion $(V_m)_{m \geq 0}$ with $(P, Q)$ that fit with the values of the $S_i$ sequence.

We define the Lucas Sequence $V_m$ such that:
\[ V_{2k+1} = S_k \tag{1} \]

Thus we have:
\[
\begin{align*}
V_2 &= S_0 = 5 \\
V_4 &= S_1 = 23 \\
V_8 &= S_2 = 527
\end{align*}
\]

If (4.2.7) page 74 ($V_{2n} = V_n^2 - 2Q^n$) applies, we have:
\[
\begin{align*}
V_4 &= V_2^2 - 2Q^2 \\
V_8 &= V_4^2 - 2Q^4
\end{align*}
\]

and thus: $Q = \frac{\sqrt{V_2^2 - V_4}}{2} = \frac{\sqrt{V_4^2 - V_8}}{2} = \pm 1$.

With (4.1.3) page 70 ($V_{n+1} = PV_n - QV_{n-1}$), and with:
\[
\begin{align*}
V_0 &= 2 \\
V_1 &= P \\
V_2 &= PV_1 - QV_0 = P^2 - 2Q
\end{align*}
\]

we have: $P = \sqrt{V_2^2 + 2Q} = \sqrt{7}$ or $\sqrt{3}$.

In the following we consider: $(P, Q) = (\sqrt{7}, 1)$.

As explained by Williams page 196, "all of the identity relations [Lucas functions] given in (4.2) continue to hold, as these are true quite without regard as to whether $P, Q$ are integers".

So, like Lehmer, we define $P = \sqrt{R}$ such that $R = 7$ and $Q = 1$ are coprime integers and we define (Property (8.4.1) page 196):
\[
\begin{align*}
V_n &= \begin{cases} 
V_n & \text{when } 2 \mid n \\
V_n/\sqrt{R} & \text{when } 2 \nmid n
\end{cases} \\
U_n &= \begin{cases} 
U_n/\sqrt{R} & \text{when } 2 \mid n \\
U_n & \text{when } 2 \nmid n
\end{cases}
\end{align*}
\]

in such a way that $V_n$ and $U_n$ are always integers.

Tables 1 to 5 give values of $U_i$, $V_i$, $\bar{U}_i \pmod{F_n}$, $\bar{V}_i \pmod{F_n}$, with $(P, Q) = (\sqrt{7}, 1)$, for $n = 1, 2, 3, 4$. 

2
2 Lehmer theorems

Like Lehmer, let define the symbols (where \((a/b)\) is the Legendre symbol):

\[
\begin{align*}
\varepsilon &= \varepsilon(p) = (D/p) \\
\sigma &= \sigma(p) = (R/p) \\
\tau &= \tau(p) = (Q/p)
\end{align*}
\]

The 2 following formulas (from page 77) will help proving properties:

\[
\text{(4.2.28)} \quad 2^{m-1}U_{mn} = \sum_{i=0}^\left\lfloor m/2 \right\rfloor \binom{m}{2i+1} D^i U^2i+1 V^{m-(2i+1)}_n
\]

\[
\text{(4.2.29)} \quad 2^{m-1}V_{mn} = \sum_{i=0}^\left\lfloor m/2 \right\rfloor \binom{m}{2i} D^i U^{2i} V^{m-2i}_n
\]

**Property (8.4.2) page 196:**

If \(p\) is an odd prime and \(p \nmid Q\), then:

\[
\begin{align*}
U_p &\equiv (D/p) \pmod{p} \\
V_p &\equiv (R/p) \pmod{p}
\end{align*}
\]

**Proof:**
Since \(p\) is a prime, and by Fermat little theorem, we have: \(2^{p-1} \equiv 1 \pmod{p}\).

- By (4.2.28), with \(m = p\) and \(n = 1\), since \(U_1 = 1\) and \(V_1 = P\), we have:

\[
2^{p-1}U_p = \sum_{i=0}^{p-1} \binom{p}{2i+1} D^i U^{2i+1}_1 V^{p-(2i+1)}_1
\]

\[
2^{p-1}U_p = \binom{p}{1} P^{p-1} + \binom{p}{3} D P^{p-3} + \ldots + \binom{p}{p} D^{p-1} P^0
\]

Since \(\binom{p}{i} \equiv 0 \pmod{p}\) when \(0 < i < p\) and \(\binom{p}{p} = 1\), we have:

\[
U_p = U_p \equiv D^{p-1}/2 \equiv (D/p) \pmod{p}
\]

- By (4.2.29), with \(m = p\) and \(n = 1\), since \(U_1 = 1\) and \(V_1 = P\), we have:

\[
2^{p-1}V_p = \sum_{i=0}^{p-1} \binom{p}{2i} D^i U^{2i}_1 V^{p-2i}_1
\]

\[
2^{p-1}V_p = \binom{p}{0} P^p + \binom{p}{2} D P^{p-2} + \ldots + \binom{p}{p-1} D^{p-1} P
\]

Since \(\binom{p}{0} = 1\), and \(\binom{p}{i} \equiv 0 \pmod{p}\) when \(0 < i < p\), we have:

\[
V_p \equiv P^p \text{ and } V_p \equiv R^{p-1}/2 \equiv (R/p) \pmod{p}
\]

\[
\square
\]
Property (8.4.3) page 197:

\[ p \text{ odd prime and } p \nmid Q \implies p \nmid \overline{U}_{p-\sigma \varepsilon} \]

Proof

By (4.2.28) with \( n = 1, V_1 = P \), since \( p \) is a prime and \( (R, Q) = 1 \), we have:

- **With: \( m = p + 1 \)**

\[ 2^{p+1}U_{p+1} = \sum_{i=0}^{\frac{p+1}{2}} \binom{p+1}{2i+1} D^i P^{p-2i} \]

\[ 2^{p+1}U_{p+1} = \begin{pmatrix} p+1 \choose 1 \end{pmatrix} P^p + \begin{pmatrix} p+1 \choose 3 \end{pmatrix} D P^{p-2} + \ldots + \begin{pmatrix} p+1 \choose p \end{pmatrix} D^{\frac{p-1}{2}} P + \begin{pmatrix} p+1 \choose p+2 \end{pmatrix} D^{\frac{p+1}{2}} P^{-1} \]

\[ 2^{p+1}U_{p+1} = (p+1)P^p + (p+1)p[\ldots] + (p+1)D^\frac{p-1}{2} P + 0D^\frac{p+1}{2} P^{-1} \]

\[ 2^{p+1}U_{p+1} = P^p + D^\frac{p-1}{2} P + p[\ldots] = P[\begin{pmatrix} p+1 \choose 3 \end{pmatrix} D^\frac{p-1}{2} + D^\frac{p+1}{2} ] + p[\ldots] \]

\[ \frac{2^{p+1}U_{p+1}}{P} = 2^{p+1}U_{p+1} \equiv R^\frac{p-1}{2} + D^\frac{p-1}{2} \equiv (R/P) + (D/p) = \sigma(p) + \varepsilon(p) \pmod{p} \]

Thus, if \( \sigma \varepsilon = \sigma(p) \times \varepsilon(p) = -1 \), then \( p \nmid U_{p+1} = U_{p-\sigma \varepsilon} \).

- **With: \( m = p - 1 \)**

\[ 2^{p-1}U_{p-1} = \sum_{i=0}^{\frac{p-1}{2}} \binom{p-1}{2i+1} D^i P^{p-2(i+1)} \]

\[ 2^{p-1}U_{p-1} = \begin{pmatrix} p-1 \choose 1 \end{pmatrix} P^{p-2} + \begin{pmatrix} p-1 \choose 3 \end{pmatrix} D P^{p-4} + \ldots + \begin{pmatrix} p-1 \choose p-2 \end{pmatrix} D^\frac{p-3}{2} P + \begin{pmatrix} p-1 \choose p \end{pmatrix} D^\frac{p-1}{2} P^{-1} \]

\[ 2^{p-1}U_{p-1} = (p-1)P^{p-2} + (p-1)DP^{p-4} + \ldots + (p-1)D^\frac{p-3}{2} P + 0D^\frac{p+1}{2} P^{-1} \]

\[ \frac{2^{p-1}U_{p-1}}{P} = -[P^{p-3} + DP^{p-5} + \ldots + D^\frac{p-3}{2}] = -\frac{P^{p-1} - D^\frac{p-1}{2}}{P^2 - D} \pmod{p} \]

\[ 2^{p-1}U_{p-1}(P^2 - D) \equiv -(P^2)^\frac{p-1}{2} + D^\frac{p-1}{2} \equiv \varepsilon(p) - \sigma(p) \pmod{p} \]

Thus, if \( \sigma \varepsilon = \sigma(p) \times \varepsilon(p) = 1 \), then \( p \nmid U_{p-1} = U_{p-\sigma \varepsilon} \). \( \square \)

Property (8.4.4) page 197

If \( p \) is an odd prime and \( p \nmid Q \), then: \( V_{p-\sigma \varepsilon} \equiv 2\sigma Q^\frac{1-\sigma \varepsilon}{2} \pmod{p} \).
Theorem 2 (8.4.1) If \( p \) is an odd prime and \( p \mid QRD \), then:

\[
\begin{cases}
    p \mid V_{p-\sigma \epsilon} & \text{when } \sigma = -\tau \\
    p \mid U_{p-\sigma \epsilon} & \text{when } \sigma = \tau
\end{cases}
\]

Definition (8.4.2) page 197 of \( \omega(m) \): For a given \( m \), denote by \( \omega = \omega(m) \) the value of the least positive integer \( k \) such that \( m \mid U_k \). If \( \omega(m) \) exists, \( \omega(m) \) is called the rank of apparition of \( m \).

Theorem 3 (8.4.3)

\[
\begin{cases}
    \text{If } k \mid n, \text{ then } U_k \mid U_n . \\
    \text{If } m \mid U_n, \text{ then } \omega(m) \mid n .
\end{cases}
\]

Theorem 4 (8.4.5) If \((m, QRD) = 1\), then \( \omega(m) \) exists.

Theorem 5 (8.4.6) If \((N, 2QRD) = 1\) and \( N \pm 1 \) is the rank of apparition of \( N \), then \( N \) is a prime.

Theorem 6 (8.4.7) If \((N, 2QRD) = 1\), \( U_{N \mp 1} \equiv 0 \pmod{N} \) and \( U_{N \mp 1} \not\equiv 0 \pmod{N} \) for each distinct prime divisor \( q \) of \( N \pm 1 \), then \( N \) is a prime.

Proof:
Let \( \omega = \omega(N) \). We see that \( \omega \mid N \pm 1 \), but \( \omega \not\mid (N \pm 1)/q \). Thus if \( q^a \parallel N \pm 1 \), then \( q^a \mid \omega \). It follows that \( \omega = N \pm 1 \) and \( N \) is a prime by Theorem 5 (8.4.6).

3 \( F_n \) prime \( \implies \) \( F_n \mid V_{\frac{F_{n-1}}{2}} \) and \( F_n \mid S_2^{n-2} \)

Let \( N = F_n = 2^{2^n} + 1 \) with \( n \geq 1 \) be an odd prime.

Let: \( P = \sqrt{R} \), \( R = 7 \), \( Q = 1 \), and \( D = P^2 - 4Q = 3 \).

Hereafter we compute \((\frac{3}{N})\) and \((\frac{7}{N})\):

- \((\frac{3}{N})\):
  
  
  Since: \[
  \begin{cases}
    N \text{ odd prime} \\
    N = (4)^{2^{n-1}} + 1 \equiv 2 \pmod{3} \\
    (\frac{N}{3}) = (\frac{2}{3}) = -1 \\
    (\frac{3}{N}) = (\frac{N}{3}) \times (-1)^{\frac{3-1}{2} \frac{N-1}{2}}
  \end{cases}
  \]

5
\( \left( \frac{7}{N} \right) : \) We have:

\[
\begin{cases}
2^3 & \equiv 1 \pmod{7} \\
2^{3a+b} & \equiv 2^b \pmod{7}
\end{cases}
\]

With \( 2^n \equiv b \pmod{3} \), we have: \( 2^{2n} + 1 \equiv 2^1 + 1 \equiv 3 \pmod{7} \). Then we study the exponents of 2, modulo 3. We have: \( 2^2 \equiv 1 \pmod{3} \), and:

\[
\begin{cases}
2^{2m} & \equiv 1 \pmod{3} \\
N & = 2^{2m} + 1 \equiv 2^1 + 1 \equiv 3 \pmod{7} \\
\left( \frac{N}{7} \right) & = \left( \frac{3}{7} \right) = -1
\end{cases}
\]

If \( n = 2m \):

\[
\begin{cases}
2^{2m+1} & \equiv 2 \pmod{3} \\
N & = 2^{2m+1} + 1 \equiv 2^1 + 1 \equiv 5 \pmod{7} \\
\left( \frac{N}{7} \right) & = \left( \frac{5}{7} \right) = -1
\end{cases}
\]

Finally, we have: \( \left( \frac{7}{N} \right) = \left( \frac{N}{7} \right)(-1)^{\frac{7-1}{2}2^{2n}} = \left( \frac{N}{7} \right) = -1 \).

So we have:

\[
\begin{cases}
\epsilon & = \left( \frac{D}{N} \right) = \left( \frac{3}{N} \right) = -1 \\
\sigma & = \left( \frac{R}{N} \right) = \left( \frac{7}{N} \right) = -1 \\
\tau & = \left( \frac{Q}{N} \right) = \left( \frac{1}{N} \right) = +1
\end{cases}
\]

Since \( \sigma = -\tau \), \( \sigma \epsilon = +1 \), and \( F_n \nmid QRD \) with \( n \geq 1 \), then by Theorem 2 (8.4.1) we have:

\[
F_n \text{ prime } \implies F_n \mid V_{2^{2n-1}} = V_{2^{2n-1}}
\]

By (1) we have: \( V_{2k-1} = S_{k-2} \) and thus, with \( k = 2^n \): \( F_n \mid S_{2^{n-2}} \).

\( \Box \)

4 \( F_n \mid S_{2^{n-2}} \implies F_n \text{ is a prime} \)

Let \( N = F_n \) with \( n \geq 1 \). By (1) we have: \( N \mid S_{2^{n-2}} \implies N \mid V_{2^{2n-1}} \).

And thus, by (4.2.6) page 74 ( \( U_{2n} = U_n V_n \) ), we have: \( N \mid U_{2^{2n-1}} \).

By (4.3.6) page 85: \( (V_n, U_n) \mid 2Q^n \) for any \( n \), and since \( Q = 1 \), then:

\( (V_{2^{2n-1}}, U_{2^{2n-1}}) = 2 \) and thus: \( N \nmid U_{2^{2n-1}} \) since \( N \) odd.

With \( \omega = \omega(N) \), by Theorem 3 (8.4.3) we have: \( \omega \mid 2^{2n} \) and \( \omega \nmid 2^{2n-1} \).

This implies: \( \omega = 2^{2n} = N - 1 \). Then \( N - 1 \) is the rank of apparition of \( N \), and thus by Theorem 5 (8.4.6) \( N \) is a prime.

\( \Box \)
This test of primality for Fermat numbers has been communicated to the
community of number theorists working on this area on mersenneforum.org
(http://www.mersenneforum.org/showthread.php?t=2130) in May 2004, and
the proof was finalized in September 2004.
Then, in a private communication, Robert Gerbicz provided a proof of the
same theorem based on $Q[\sqrt{21}]$.

5 Numerical Examples

\[(\text{mod } F_2)\] $S_0 = 5 \mapsto 6 \mapsto S_{2^2-2} \equiv 0$

\[(\text{mod } F_3)\] $S_0 = 5 \mapsto 23 \mapsto 13 \mapsto 167 \mapsto 131 \mapsto 197 \mapsto -60 \mapsto S_{2^3-2} \equiv 0$

\[(\text{mod } F_4)\] $S_0 = 5 \mapsto 23 \mapsto 527 \mapsto 15579 \mapsto 21728 \mapsto 42971 \mapsto 1864 \mapsto 1033 \mapsto 18495 \mapsto 27420 \mapsto 15934 \mapsto 960 \mapsto 4080 \mapsto S_{2^4-2} \equiv 0$

6 Appendix: Table of $U_i$ and $V_i$

With $n = 2, 3, 4$, we have the following (not proven) properties (modulo $F_n$):

\[
\begin{align*}
U_{F_n-5} &\equiv 5 \\
U_{F_n-4} &\equiv 6 \\
U_{F_n-3} &\equiv 1 \\
U_{F_n-2} &\equiv 1 \\
U_{F_n-1} &\equiv 0 \\
U_{F_n} &\equiv -1 \\
U_{F_n+1} &\equiv -1 \\
U_{F_n+2} &\equiv -6 \\
U_{F_n+3} &\equiv -5 \\
\end{align*}
\]

\[
\begin{align*}
V_{F_n-5} &\equiv -23 \\
V_{F_n-4} &\equiv -4 \\
V_{F_n-3} &\equiv -5 \\
V_{F_n-2} &\equiv -1 \\
V_{F_n-1} &\equiv -2 \\
V_{F_n} &\equiv -1 \\
V_{F_n+1} &\equiv -5 \\
V_{F_n+2} &\equiv -4 \\
V_{F_n+3} &\equiv -23 \\
\end{align*}
\]

The values of $U_n$ and $V_n$ ($n \geq 1$) with $(P, Q) = (\sqrt{3}, -1)$ can be built by:

\[
\begin{align*}
\bar{U}_{2n} &= U_{2n} \\
\bar{U}_{2n+1} &= V_{2n+1} \\
\bar{V}_{2n} &= \bar{V}_{2n} \\
\bar{V}_{2n+1} &= \bar{U}_{2n+1} \\
\end{align*}
\]

Values of $U_i$ and $V_i$ in previous tables can be computed easily by the following
PARI/gp programs:

\[
U_{2j+1} : U_0=1; U_1=6; \text{ for}(i=1,N, U_0=5*U_1-U_0; U_1=5*U_0-U_1; \text{ print}(4*i+1," ",U_0); \text{ print}(4*i+1," ",U_1))
\]
\[
\begin{array}{c|c|c}
 i & U_i & V_i \\
 \hline
 0 & 0 \times \sqrt{7} & 2 \\
 1 & 1 & 1 \times \sqrt{7} \\
 2 & 1 \times \sqrt{7} & 5 \\
 3 & 6 & 4 \times \sqrt{7} \\
 4 & 5 \times \sqrt{7} & 23 \\
 5 & 29 & 19 \times \sqrt{7} \\
 6 & 24 \times \sqrt{7} & 110 \\
 7 & 139 & 91 \times \sqrt{7} \\
 8 & 115 \times \sqrt{7} & 527 \\
 9 & 666 & 436 \times \sqrt{7} \\
 10 & 551 \times \sqrt{7} & 2525 \\
 11 & 3191 & 2089 \times \sqrt{7} \\
 12 & 2640 \times \sqrt{7} & 12098 \\
 13 & 15289 & 10009 \times \sqrt{7} \\
 14 & 12649 \times \sqrt{7} & 57965 \\
 15 & 73254 & 47956 \times \sqrt{7} \\
 16 & 60605 \times \sqrt{7} & 277727 \\
 17 & 350981 & 229771 \times \sqrt{7} \\
 18 & 290376 \times \sqrt{7} & 1330670 \\
 19 & 1681651 & 1100899 \times \sqrt{7} \\
 20 & 1391275 \times \sqrt{7} & 6375623 \\
 21 & 8057274 & 5274724 \times \sqrt{7} \\
 22 & 6665999 \times \sqrt{7} & 30547445 \\
 23 & 38604719 & 25272721 \times \sqrt{7} \\
 24 & 31938220 \times \sqrt{7} & 146361002 \\
 25 & 184966321 & 121088881 \times \sqrt{7} \\
 26 & 153027601 \times \sqrt{7} & 701260565 \\
 27 & 886226886 & 580171684 \times \sqrt{7} \\
 28 & 733199285 \times \sqrt{7} & 3359941223 \\
 29 & 4246168109 & 2779769539 \times \sqrt{7} \\
 30 & 3512968824 \times \sqrt{7} & 1609845550 \\
 31 & 20344613659 & 13318676011 \times \sqrt{7} \\
 32 & 16831644835 \times \sqrt{7} & 77132286527 \\
 33 & 97476900186 & 63813610516 \times \sqrt{7} \\
 34 & 80645255351 \times \sqrt{7} & 369562987085 \\
 35 & 467039887271 & 305749376569 \times \sqrt{7} \\
 36 & 386394631920 \times \sqrt{7} & 1770682648889 \\
 37 & 2237722536169 & 146493272329 \times \sqrt{7} \\
 38 & 1851327904249 \times \sqrt{7} & 8483850257405 \\
 39 & 10721572793574 & 7018916985076 \times \sqrt{7} \\
 40 & 8870244889325 \times \sqrt{7} & 40648568638127 \\
\end{array}
\]

Table 1: \( P = \sqrt[8]{7} \), \( Q = 1 \)
Table 2: $P = \sqrt{7}$, $Q = 1$, Modulo $F_1$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\overline{U}_i \pmod{F_1}$</th>
<th>$\overline{V}_i \pmod{F_1}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
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<td>4</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
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<td>4</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3: $P = \sqrt{7}$, $Q = 1$, Modulo $F_2$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\overline{U}_i \pmod{F_2}$</th>
<th>$\overline{V}_i \pmod{F_2}$</th>
</tr>
</thead>
<tbody>
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<td>2</td>
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<td>1</td>
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<td>8</td>
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Table 4: \( P = \sqrt{7} \), \( Q = 1 \), Modulo \( F_3 \)
Table 5: \( P = \sqrt{7} \), \( Q = 1 \), Modulo \( F_4 \)

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