A very simple property of Mersenne numbers

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Let show that:  $2^q - 1 = 1 + 6(1 + 2^2 + 2^4 + 2^6 + \dots + 2^{2\frac{q-1}{2}})$ . Which is equivalent to :

$$2^{q} - 1 = 1 + 6 \sum_{i=0}^{\frac{q-1}{2}} (2^{i})^{2} = 1 + 2 \times 3 \sum_{i=0}^{\frac{q-1}{2}} 2^{2i} \quad (I)$$

First :  $2^q - 1 = 2^q - 2 + 1 = 2(2^{q-1} - 1) + 1$ .

Since q is an odd number : q - 1 = 2n.

Now, show:

$$2^{2n} - 1 = 3\sum_{i=0}^{n-1} 2^{2i} \quad (II)$$

With n = 1, we have:  $2^{2n} - 1 = 3 = 3 \times 1 = 3(2^{2 \times 0})$ . With n = 2, we have:  $2^{2n} - 1 = 15 = 3 \times 5 = 3(1 + 2^{2 \times 1}) = 3(1 + 2^{2 \times (n-1)})$ . Thus, property (*II*) is true for ranks n = 1 and 2.

Suppose that property (II) is true at rank n.

Then, at rank n+1, we have:  $2^{2(n+1)}-1=4\times 2^{2n}-1=2^{2n}-1+3\times 2^{2n}$  . And thus:

$$2^{2(n+1)} - 1 = 3\sum_{i=0}^{n-1} 2^{2i} + 3 \times 2^{2n} = 3(2^{2\times 0} + 2^{2\times 1} + \dots + 2^{2\times (n-1)} + 2^{2\times n}) = 3\sum_{i=0}^{n} 2^{2i} + 3 \times 2^{2n} = 3(2^{2\times 0} + 2^{2\times 1} + \dots + 2^{2\times (n-1)} + 2^{2\times n}) = 3\sum_{i=0}^{n} 2^{2i} + 3 \times 2^{2n} = 3(2^{2\times 0} + 2^{2\times 1} + \dots + 2^{2\times (n-1)} + 2^{2\times n}) = 3\sum_{i=0}^{n} 2^{2i} + 3 \times 2^{2n} = 3(2^{2\times 0} + 2^{2\times 1} + \dots + 2^{2\times (n-1)} + 2^{2\times n}) = 3\sum_{i=0}^{n} 2^{2i} + 3 \times 2^{2n} = 3(2^{2\times 0} + 2^{2\times 1} + \dots + 2^{2\times (n-1)} + 2^{2\times n}) = 3\sum_{i=0}^{n} 2^{2i} + 3 \times 2^{2n} =$$

Which shows that the property (II) is also true at rank n+1 and thus that it is true for any n > 0. And so the property (I) is true for any odd q > 2.

As an example:  $2^{11} - 1 = 1 + 2((2^5)^2 - 1) = 1 + 2 \times 3(1 + 2^2 + 4^2 + 8^2 + 16^2)$ which produces a nice figure: a square of side 1 + 2 identical squares of side  $2^5$  missing a square of side 1 each. And each of the *nearly squares* is made of 3 times a set of squares of side  $(2^i)^2$  with i = 0..4.