Generalized Pell Numbers

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This paper presents a generalization of Pell numbers. With no proof.

1 General Properties

1.1 Definitions

Let define:

$$G_n[m,r] = \sum_{\substack{0 \le k \le n \\ k \equiv r \pmod m}} \binom{n}{k} m^{\frac{k-r}{m}} = \sum_{i=0}^{\left\lfloor \frac{n-r}{m} \right\rfloor} \binom{n}{mi+r} m^i$$

We have the recursion:

$$G_n[m,r] = \sum_{i=1}^{m-1} \binom{m}{i} (-1)^{i+1} G_{n-i}[m,r] + (m + (-1)^{m+1}) G_{n-m}[m,r]$$

and the limit:

$$\lim_{n \to \infty} \frac{G_{n+1}[m,r]}{G_n[m,r]} = 1 + \sqrt[m]{m}$$

with:

$$0 \le n \le m \ , \ 0 \le r < m \ , \ G_n[m,r] = \binom{n}{r} \ , \text{ except: } : G_m[m,0] = m+1$$

Let define the following polynomial:

$$\mathfrak{P}_m(X) = \sum_{i=0}^{m-1} \binom{m}{i} (-1)^i X^{m-i} - (m + (-1)^{m+1}) = (X-1)^m - m$$

Its discriminant is: $(-1)^{(m-(m \bmod 2))/2} m^{2m-1}$ and: $1 + \sqrt[m]{m}$ is a root.

1.2 $G_n[m,0]$ modulo a prime

With: $M = 2^m$, $G_n[M,0] \bmod (M+1)$ gives interesting results:
For $m = 2, 4, 8$, but NOT for $m = 3, 5, 6, 7, 9$, we have: $G_n[M,0] \bmod (M+1) = 1$ for all n except for $n = M, 2M, 2M+1, 3M, 3M+1, 3M+2, ..., kM + (0..k-1)$. (Hereafter we note: $G_n[m,0] = G_n[m]$.)

$G_{1\times4+0}[4] \bmod 5 = G_{1\times16+0}[16] \bmod 17 = G_{1\times256+0}[256] \bmod 257 = 0$
$G_{2\times4+0}[4] \bmod 5 = G_{2\times16+0}[16] \bmod 17 = G_{2\times256+0}[256] \bmod 257 = 2$
$G_{2\times4+1}[4] \bmod 5 = G_{2\times16+1}[16] \bmod 17 = G_{2\times256+1}[256] \bmod 257 = -1$
$G_{3\times4+0}[4] \bmod 5 = G_{3\times16+0}[16] \bmod 17 = G_{3\times256+0}[256] \bmod 257 = 0$
$G_{3\times4+1}[4] \bmod 5 = G_{3\times16+1}[16] \bmod 17 = G_{3\times256+1}[256] \bmod 257 = 5$

$G_{3\times 4+2}[4] \bmod 5 = G_{3\times 16+2}[16] \bmod 17 = G_{3\times 256+2}[256] \bmod 257 = -3$
$G_{4\times 4+0}[4] \bmod 5 = G_{4\times 16+0}[16] \bmod 17 = G_{4\times 256+0}[256] \bmod 257 = 2$
$G_{4\times 4+1}[4] \bmod 5 = G_{4\times 16+1}[16] \bmod 17 = G_{4\times 256+1}[256] \bmod 257 = -5$
$G_{4\times 4+2}[4] \bmod 5 = G_{4\times 16+2}[16] \bmod 17 = G_{4\times 256+2}[256] \bmod 257 = 13$
$G_{4\times 4+3}[4] \bmod 5 = G_{4\times 16+3}[16] \bmod 17 = G_{4\times 256+3}[256] \bmod 257 = -7$

The same properties seem to appear when M is any number such that $M+1$ is prime. Checked for $M = 6, 10, 12, 18, 22$.

2 Examples

2.1 $m = 1$

With $m = 1$ and $r = 0$, $G_n[1, 0]$ gives the powers of 2:

$$2G_n[1, 0] = PG_{n-1}[1, 0] = 2^n \ , \quad \text{with: } P = 2$$

The polynomial is: $\mathfrak{P}(x) = x - P = x - 2 = (x - 1)^1 - 1$, which discriminant is: $D = 1 = 1^{2\times 1-1}$ and has 1 root: $\alpha = 1 + \sqrt[1]{1} = 2$. And we have: $\alpha = P$.

2.2 $m = 2$

With $m = 2$ and $r = 0$ or 1, $G_n[2, r]$ gives the Pell numbers: $(P, Q) = (2, -1)$

$$G_n[2, r] = PG_{n-1}[2, r] - QG_{n-2}[2, r] = 2G_{n-1}[2, r] + G_{n-2}[2, r]$$

$$2G_n[2, 0] = V_n(P, Q) = PV_n(P, Q) - QV_n(P, Q)$$

$$G_n[2, 1] = U_n(P, Q) = PU_n(P, Q) - QU_n(P, Q)$$

The polynomial is: $\mathfrak{P}(x) = x^2 - Px + Q = x^2 - 2x - 1 = (x - 1)^2 - 2$, which discriminant is: $D = P^2 - 4Q = 8 = 2^{2\times 2-1}$ and has 2 roots: $\alpha = 1 + \sqrt[2]{2}$ and $\beta = 1 - \sqrt[2]{2}$. And we have: $\alpha + \beta = P$, $\alpha\beta = Q$, $\alpha - \beta = \sqrt{D} = 2\sqrt{2}$. $U_n(2, -1) = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $V_n(2, -1) = \alpha^n + \beta^n$.

2.3 $m = 3$

With $m = 3$ and $r = 0, 1$ or 2, $G_n[3, r]$ gives: $(P, Q, R) = (3, 3, 4)$

$$G_n[3, r] = PG_{n-1}[3, r] - QG_{n-2}[3, r] + RG_{n-3}[3, r]$$

$$G_n[3, r] = 3G_{n-1}[3, r] - 3G_{n-2}[3, r] + 4G_{n-3}[3, r]$$

The polynomial is: $\mathfrak{P}(x) = x^3 - Px^2 + Qx - R = x^3 - 3x^2 + 3x - 4 = (x - 1)^3 - 3$, which discriminant is: $D = P^2Q^2 + 18PQR - 4Q^3 - 4P^3R - 27R^2 = -243 = -3^{2\times 3-1}$ and has 3 roots: $\alpha = 1 + \sqrt[3]{3}$, $\beta = 1 - (\sqrt[3]{3} - i\sqrt[6]{3^5})/2$, and $\gamma = 1 - (\sqrt[3]{3} + i\sqrt[6]{3^5})/2$. And we have: $\alpha + \beta + \gamma = P$, $\alpha\beta + \beta\gamma + \gamma\alpha = Q$, and $\alpha\beta\gamma = R$. Let define: $A_n = \alpha^n + \beta^n + \gamma^n$. We have: $3G_n[3, 0] = A_n$.

m	r	0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
2	0	1	1	3	7	17	41	99	239	577	1393	3363	8119	19601
2	1	0	1	2	5	12	29	70	169	408	985	2378	5741	13860
3	0	1	1	1	4	13	31	70	169	421	1036	2521	6139	14998
3	1	0	1	2	3	7	20	51	121	290	711	1747	4268	10407
3	2	0	0	1	3	6	13	33	84	205	495	1206	2953	7221
4	0	1	1	1	1	5	21	61	141	297	649	1561	3961	9965
4	1	0	1	2	3	4	9	30	91	232	529	1178	2739	6700
4	2	0	0	1	3	6	10	19	49	140	372	901	2079	4818
4	3	0	0	0	1	4	10	20	39	88	228	600	1501	3580
5	0	1	1	1	1	1	6	31	106	281	631	1286	2586	5611
5	1	0	1	2	3	4	5	11	42	148	429	1060	2346	4932
5	2	0	0	1	3	6	10	15	26	68	216	645	1705	4051
5	3	0	0	0	1	4	10	20	35	61	129	345	990	2695
5	4	0	0	0	0	1	5	15	35	70	131	260	605	1595
6	0	1	1	1	1	1	1	7	43	169	505	1261	2773	5581
6	1	0	1	2	3	4	5	6	13	56	225	730	1991	4764
6	2	0	0	1	3	6	10	15	21	34	90	315	1045	3036
6	3	0	0	0	1	4	10	20	35	56	90	180	495	1540
6	4	0	0	0	0	1	5	15	35	70	126	216	396	891
6	5	0	0	0	0	0	1	6	21	56	126	252	468	864
7	0	1	1	1	1	1	1	1	8	57	253	841	2311	5545
7	1	0	1	2	3	4	5	6	7	15	72	325	1166	3477
7	2	0	0	1	3	6	10	15	21	28	43	115	440	1606
7	3	0	0	0	1	4	10	20	35	56	84	127	242	682
7	4	0	0	0	0	1	5	15	35	70	126	210	337	579
7	5	0	0	0	0	0	1	6	21	56	126	252	462	799
7	6	0	0	0	0	0	0	1	7	28	84	210	462	924
8	0	1	1	1	1	1	1	1	1	9	73	361	1321	3961
8	1	0	1	2	3	4	5	6	7	8	17	90	451	1772
8	2	0	0	1	3	6	10	15	21	28	36	53	143	594
8	3	0	0	0	1	4	10	20	35	56	84	120	173	316
8	4	0	0	0	0	1	5	15	35	70	126	210	330	503
8	5	0	0	0	0	0	1	6	21	56	126	252	462	792
8	6	0	0	0	0	0	0	1	7	28	84	210	462	924
8	7	0	0	0	0	0	0	0	1	8	36	120	330	792

Table 1: First values of $G_n[m,r], n = 1..12, r = 0..m-1, m = 1..8$

2.4 Values of $G_n[m, r]$ for $m = 1..8$

Table 1 provides the first 12 values for m = 1 to m.

3 A proof of the recursion by Zhi-Wei SUN

As the characteristic function of $k = r \pmod m$ equals:

$$m^{-1} \sum_{j=0}^{m-1} e^{2\pi i j(k-r)/m} \ ,$$

we have:

$$G_n[m, r] = m^{-1} \sum_{j=0}^{m-1} m^{-r/m} e^{-2\pi i j r/m} (1 + m^{1/m} e^{2\pi i j/m})^n \ .$$

Thus $G_n[m, r]$ is a recurrent sequence (with respect to n) of order m with the characteristic polynomial:

$$P(x) = \prod_{j=0}^{m-1} (x - 1 - m^{1/m} e^{2\pi i j/m}) = (x-1)^m - m \ .$$

So $G_n[m, r]$ satisfies the recursion you described.

A more detailed explanation:

If $P(x) = (x - \alpha_1)...(x - \alpha_m) = x^m - a_1 x^{m-1} - ... - a_m$, then

$$u_n = \sum_{j=1}^{m} c_j * \alpha_j^n$$

satisfies the recursion

$$u_n = \sum_{i=1}^{m} a_i u_{n-i} \quad (\text{for } n \geq m).$$

In fact,

$$\sum_{i=1}^{m} a_i u_{n-i} = \sum_{i=1}^{m} a_i \sum_{j=1}^{m} c_j \alpha_j^{n-i} = \sum_{j=1}^{m} c_j \alpha_j^{n-m} \sum_{i=1}^{m} a_i \alpha_j^{m-i}$$

$$= \sum_{j=1}^{m} c_j \alpha_j^{n-m} (\alpha_j^m - P(\alpha_j)) = u_n \ .$$